Collision operator of GYSELA

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Abstract

The collision operator used in GYSELA is described in this note. For more details please read the associated paper [1]

1. Presentation of the model

The linearized collisional operator describing the collisions of species a colliding on species b takes the form

$$C_{ab}(F_a, F_b) = C^0_{ab}(F_{M0a}, F_{M0b}) + C^1_{ab}(F_a, F_b),$$

where F_{M0a} represents the local unshifted Maxwellian with density n_a and temperature T_a

$$F_{M0a}(\boldsymbol{x}, \boldsymbol{v}, t) = n_a(\boldsymbol{x}, t) \left(\frac{1}{2\pi v^2_{Ta}}\right)^{3/2} \exp\left(-x_a^2\right).$$

A normalized speed has been used $x_a = \frac{v}{\sqrt{2} v_{Ta}}$,with $v_{Ta} = \sqrt{\frac{T_a}{m_a}}$ the thermal velocity.

C^0_{ab} represents the exchange of energy between the unshifted Maxwellians

$$C^0_{ab}(F_{M0a}, F_{M0b}) = \frac{T_b - T_a}{T_b} x_a^2 \nu_{E,ab} F_{M0a}$$

Neglecting all finite Larmor radius effects, C^1_{ab} is composed of three terms

$$C^1_{ab}(F_a, F_b) = C_{v,ab}(F_a) + C_{d,ab}(F_a) + C_{\parallel,ab}(F_a, F_b)$$

$C_{v,ab}$ is an operator acting on the norm of the velocity. When written in the set of variables $\left(v_\parallel, v_\perp\right)$, it reads as follows:

$$\begin{aligned} C_{v,ab}\left(F_a\right) &= \frac{1}{2v_\perp}\frac{\partial}{\partial v_\perp}\left[F_{M0a}\nu_{v,ab}v_\perp^2\left(v_\perp\frac{\partial g_{ab}}{\partial v_\perp} + v_\parallel\frac{\partial g_{ab}}{\partial v_\parallel}\right)\right] \\ &+ \frac{1}{2}\frac{\partial}{\partial v_\parallel}\left[F_{M0a}\nu_{v,ab}v_\parallel\left(v_\perp\frac{\partial g_{ab}}{\partial v_\perp} + v_\parallel\frac{\partial g_{ab}}{\partial v_\parallel}\right)\right] \end{aligned}$$

$C_{d,ab}$ modifies the direction of the velocity vector (deflection)

$$\begin{aligned} C_{d,ab}\left(F_a\right) &= \frac{1}{2v_\perp}\frac{\partial}{\partial v_\perp}\left[F_{M0a}\nu_{d,ab}v_\perp v_\parallel\left(v_\parallel\frac{\partial g_{ab}}{\partial v_\perp} - v_\perp\frac{\partial g_{ab}}{\partial v_\parallel}\right)\right] \\ &+ \frac{1}{2}\frac{\partial}{\partial v_\parallel}\left[F_{M0a}\nu_{d,ab}v_\perp\left(-v_\parallel\frac{\partial g_{ab}}{\partial v_\perp} + v_\perp\frac{\partial g_{ab}}{\partial v_\parallel}\right)\right] \end{aligned}$$

Finally the term $C_{\parallel,ab}$ ensures momentum exchange between species and the conservation of the total parallel momentum.

$$C_{\parallel,ab}\left(F_a, F_b\right) = -\frac{\nu_{s,ab}(v)}{v^2_{Ta}} v_\parallel \left(U_{\parallel d,a} - U_{\parallel ba}\right) F_{M0a}$$

The normalized distribution function has to be shifted to ensure that $C_{v,ab}$ and $C_{d,ab}$ conserve momentum and energy

$$g_{ab} = f_a - \frac{v_\parallel U_{\parallel d,a}}{v^2_{Ta}} - x_a^2 q_{ba} \text{ with } f_a = \frac{F_a}{F_{M0a}}$$

More specifically, $U_{\parallel d,a}$ ensures that $C_{v,ab}$ and $C_{d,ab}$ conserve momentum.

$$\frac{v}{v_{Ta}^2} U_{\parallel d,a}(v) = \frac{3}{2} \int d\xi \xi f_a \text{ with } \xi = \frac{v_\parallel}{v}$$

Then in order to take into account momentum exchange between species while keeping the total momentum constant, a second velocity $U_{\parallel ab}$ is chosen as

$$U_{\parallel ab} = \frac{\left\langle \nu_{s,ab} v^2 U_{\parallel d,a} \right\rangle_a}{\left\langle \nu_{s,ab} v^2 \right\rangle_a}$$

A dimensionless parameter q_{ab} accounting for energy exchange between species is defined as

$$q_{ab} = T_b \frac{\left\langle \nu_{E,ab} \frac{m_a v^2}{2} f_a \right\rangle_a}{\left\langle \nu_{E,ab} \left(\frac{m_a v^2}{2} \right)^2 \right\rangle_a}$$

The bracket corresponds to mean values in velocity space $< ... >= \int d^3\boldsymbol{v} \frac{F_{M0a}}{n_a} ...$. Different frequencies appear in the previous expressions. They are defined as follow :

- the Hirshman and Sigmar's inter-species collision frequency

$$\nu_{ab}^{HS} = \sqrt{2} \frac{N_b Z_b^2}{N_a Z_a^2} \nu_{aa}$$

- the velocity modulus diffusion rate

$$\nu_{v,ab}(x_a) = \nu_{ab}^{HS} x_{ba} \frac{\Theta(x_b)}{x_a^2}$$

- the deflection frequency

$$\nu_{d,ab}(x_a) = \nu_{ab}^{HS} x_{ba} \frac{\Psi(x_b)}{x_a^2}$$

- the slowing-down frequency

$$\nu_{s,ab} = \nu_{ab}^{HS} \left(1 + \frac{m_a}{m_b} \right) x_{ba}^3 \Theta(x_b)$$

- the energy-loss rate is defined as

$$\nu_{E,ab} = -\frac{1}{v^4 F_{M0a}} \frac{\partial}{\partial v} \left(\nu_{v,ab} F_{M0a} v^5 \right)$$

Where the ratio between the thermal velocities is introduced $x_{ba} = \frac{v_{Ta}}{v_{Tb}}$. We also define the following functions

$$\Psi(x) = \frac{3\sqrt{\pi}}{4} \frac{1}{x} \left[\Phi(x) - G(x) \right]$$

$$\Theta(x) = \frac{3\sqrt{\pi}}{2} \frac{G(x)}{x}$$

$$G(x) = \frac{1}{2x^2} \left[\Phi(x) - x\Phi'(x) \right]$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dy \exp\left(-y^2\right)$$

The function Φ is the error function and G is the Chandrasekhar function.

2. Numerical implementation of the collision operator

2.1. Separation of the different collision terms

The collision operator is difficult to treat as a whole. It is much easier to split the different parts of the operator and treat them separately with a time splitting scheme. The time splitting is the following:

$$\begin{cases} \frac{\partial F_{M0a}}{\partial t} = \sum_b C^0_{ab}\left(F_{M0a}, F_{M0b}\right) & (\Delta t/2) \\ \frac{\partial F_a}{\partial t} = \sum_b C_{\parallel,ab}\left(F_a, F_b\right) & (\Delta t/2) \\ \frac{\partial F_a}{\partial t} = \sum_b C_{v,ab}\left(F_a, F_b\right) + C_{d,ab}\left(F_a, F_b\right) & (\Delta t) \\ \frac{\partial F_a}{\partial t} = \sum_b C_{\parallel,ab}\left(F_a, F_b\right) & (\Delta t/2) \\ \frac{\partial F_{M0a}}{\partial t} = \sum_b C^0_{ab}\left(F_{M0a}, F_{M0b}\right) & (\Delta t/2) \end{cases} \tag{1}$$

In the following, we describe the three

2.2. Evolution due to C^0_{ab}

The term

$$\frac{\partial F_{M0a}}{\partial t} = \sum_b C^0_{ab}\left(F_{M0a}, F_{M0b}\right)$$

is equivalent to

$$\frac{\partial T_a}{\partial t} = \sum_b m_a \nu_{ab}\left[\frac{2}{m_a+m_b}\left(T_b - T_a\right) + \frac{2}{3}V_{\parallel a}\left(V_{\parallel a} - V_{\parallel b}\right)\right] \tag{2}$$

2.3. Evolution due to $C_{\parallel,ab}$

$$\begin{aligned} C_{\parallel,ab} &= \nu_{s,ab}\frac{m_a}{T_a}v_\parallel F_{M0a} \\ &\times \left[V_{\parallel b} - V_{\parallel a} + \frac{q_{\parallel a}}{n_a T_a}\left(1 - \frac{2}{5}x_a^2\right) - \frac{3}{5}\frac{q_{\parallel b}}{n_b T_b}\left(\frac{1}{1+x_{ba}^2}\right)\right] \end{aligned} \tag{3}$$

2.4. Evolution due to $C_{v,ab} + C_{d,ab}$

This is by far the most difficult part of the collision operator. We use the fact that

$$\frac{\partial F_a}{\partial t} = \sum_b C_{v,ab}\left(F_a, F_b\right) + C_{d,ab}\left(F_a, F_b\right) \Leftrightarrow \frac{\partial f_a}{\partial t} = \sum_b \bar{C}_{ab} \tag{4}$$

where we have defined the normalized collision operator $\bar{C}_{ab}(F) = \frac{C_{v,ab}(F)+C_{d,ab}(F)}{F_{M0a}}$. This approach is possible because a Maxwellian is in the kernel of $C_{v,ab} + C_{d,ab}$.

The derivatives with respect to $v_\perp$ are treated by projecting the distribution in this direction on an orthogonal basis (Laguerre polynomials), then making evolve the projection coefficients and finally coming back on the distribution function. The reason for this choice is the relatively low number of grid points in this direction (typically between 32 and 64). The first step is therefore to compute the projection of the normalized distribution function

$$f_a(\boldsymbol{r}, v_\parallel, u, t) = \sum_l \alpha_{\ell,a}(\boldsymbol{r}, v_\parallel, t) P_\ell(u)$$

where Laguerre polynomials are chosen. The associated scalar product is

$$\langle f|g\rangle = \int_0^\infty dx e^{-x} f\left(x\right) g\left(x\right)$$

Therefore one has

$$\alpha_{\ell,a}(\boldsymbol{r}, v_\parallel, t) = \langle f_a|P_\ell\rangle = \int_0^\infty du e^{-u} f_a\left(u\right) P_\ell\left(u\right)$$

where $u = \frac{\mu B}{T}$

We then define

$$\begin{cases} \alpha'_{i,a} = \alpha_{i,a} + \kappa_{i,a} & \text{if } i < 2 \\ \alpha'_{i,a} = \alpha_{i,a} & \text{otherwise} \end{cases}$$

with

$$\kappa_{0,a} = -\left\{ \frac{m_a v_\parallel}{T_a} \left[V_{\parallel a} - \frac{q_{\parallel a}}{n_a T_a} \left(\frac{3}{5} - \frac{m_a v_\parallel^2}{5 T_a} \right) \right] \right\}$$

and

$$\kappa_{1,a} = \frac{2 m_a v_\parallel q_{\parallel a}}{5 n_a T_a^2}$$

The evolution of the coefficients is given by

$$\frac{\partial \alpha_{\ell,a}}{\partial t} = \sum_j \left[\alpha'_{j,a} N_{0,a}^{jl} + \frac{\partial \alpha'_{j,a}}{\partial v_\parallel} N_{1,a}^{jl} + \frac{\partial^2 \alpha'_{j,a}}{\partial v_\parallel^2} N_{2,a}^{jl} \right] \tag{5}$$

with

$$\begin{aligned} N_{0,a}^{jl} &= j \sum_b \sum_{i=0}^{j+\ell} C_{j\ell}^i \begin{bmatrix} -2L_{i,ab}^{(0)} + 2L_{i,ab}^{(1)} + \left(3 - \frac{v_\parallel^2}{v_{Ta}^2}\right) L_{i,ab}^{(2)} \\ -2L_{i+1,ab}^{(2)} + \left(2 + \frac{v_\parallel^2}{v_{Ta}^2}\right) L_{i,ab}^{(4)} \end{bmatrix} \\ &- j \sum_b \sum_{i'=0}^{j+\ell-1} C_{j-1,\ell}^{i'} \begin{bmatrix} 2L_{i',ab}^{(1)} + \left(3 - \frac{v_\parallel^2}{v_{Ta}^2}\right) L_{i',ab}^{(2)} \\ + \left(2 + \frac{v_\parallel^2}{v_{Ta}^2}\right) L_{i',ab}^{(4)} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} N_{1,a}^{jl} &= v_\parallel \sum_b \sum_{i=0}^{j+\ell} C_{j\ell}^i \begin{bmatrix} -L_{i,ab}^{(0)} + L_{i,ab}^{(1)} + \left(2 + 2j - \frac{v_\parallel^2}{2v_{Ta}^2}\right) L_{i,ab}^{(2)} \\ -L_{i+1,ab}^{(2)} + L_{i,ab}^{(3)} \end{bmatrix} \\ &- v_\parallel \sum_b \sum_{i'=0}^{j+\ell-1} C_{j-1,\ell}^{i'} 2j L_{i',ab}^{(2)} \end{aligned}$$

$$N_{2,a}^{jl} = \sum_b \sum_{i=0}^{j+\ell} C_{j\ell}^i v_{Ta}^2 \left[L_{i,ab}^{(0)} + \frac{v_\parallel^2}{2v_{Ta}^2} L_{i,ab}^{(2)} \right]$$

where the coefficients $C_{j\ell}^i$ are defined as

$$P_j(u) P_\ell(u) = \sum_{i=0}^{j+\ell} C_{j\ell}^i u^i$$

It is possible to show that

$$\begin{cases} L_{0,ab}^{(1)} = & -\frac{D_{d,ab}\left(v_\parallel, u=0\right)}{v_{Ta}^2} + L_{0,ab}^{(0)} \\ L_{i,ab}^{(1)} = & L_{i,ab}^{(0)} - i L_{i-1,ab}^{(0)} \text{ for } i > 0 \end{cases}$$

$$\begin{cases} L_{0,ab}^{(3)} = & \frac{v_\parallel^2}{2v_{Ta}^2} \left[L_{0,ab}^{(2)} - \left(\nu_{v,ab} - \nu_{d,ab}\right) \left(v_\parallel, u=0\right) \right] + L_{1,ab}^{(2)} - L_{0,ab}^{(2)} \\ L_{i,ab}^{(3)} = & \frac{v_\parallel^2}{2v_{Ta}^2} \left[L_{i,ab}^{(2)} - i L_{i-1,ab}^{(2)} \right] + L_{i+1,ab}^{(2)} - (i+1) L_{i,ab}^{(2)} \text{ for } i > 0 \end{cases}$$

$$\begin{cases} L_{0,ab}^{(4)} = & L_{0,ab}^{(2)} - \left(\nu_{v,ab} - \nu_{d,ab}\right) \left(v_\parallel, u=0\right) \\ L_{i,ab}^{(3)} = & L_{i,ab}^{(2)} - i L_{i-1,ab}^{(2)} \text{ for } i > 0 \end{cases}$$

Up to now, no approximation was performed. But one still need to compute $L_{i,ab}^{(0)}$ and $L_{i,ab}^{(2)}$. A good proxy for these quantities is:

$$L_{i,ab}^{(0)} = 0.75 \sqrt{\pi} \nu_{ab}^{HS} I_i^{(0)} \left(\frac{v_\parallel^2}{2v_{Ta}^2} + \frac{9\pi}{16} \frac{v_{Tb}^2}{v_{Ta}^2} \right)$$

$$L^{(2)}_{i,ab} = -0.75\sqrt{\pi}\nu^{HS}_{ab} I^{(1)}_i \left(\frac{v^2_\parallel}{2v^2_{Ta}} + 2.1\frac{v^2_{Tb}}{v^2_{Ta}} \right)$$

With

$$I^{(n)}_i(x) = \int_0^\infty du \frac{u^i e^{-u}}{(u+x)^{n+1/2}}$$

It can be shown that

$$I^{(n)}_i(x) = e^x \sum_{k=0}^{i} \left(\begin{array}{c} i \\ k \end{array} \right) (-x)^{i-k} J_{k-n}(x)$$

where

$$J_i(x) = \int_x^\infty du\ e^{-u} u^{i-1/2}$$

which can be easily computed by recurrence.

2.5. Numerical scheme for evolution in time

An explicit scheme is used to compute the evolution of the distribution function due to C^0_{ab} and $C_{\parallel ab}$. Eq.(5) is solved with a Crank-Nicholson scheme for stability reasons. The resolution of the Crank-Nicholson is detailed here : the problem can be written in a vectorized form

$$\frac{\partial \boldsymbol{\alpha}}{\partial \hat{t}} = T\boldsymbol{\alpha} + \boldsymbol{S}$$

with

$$\boldsymbol{\alpha} = \left(\begin{array}{c} \vdots \\ \alpha^{(k)}_0 \\ \alpha^{(k)}_1 \\ \vdots \\ \alpha^{(k)}_{Npol-1} \\ \alpha^{(k+1)}_0 \\ \vdots \end{array} \right)$$

and $0 \leq k \leq k_{max}$,

$$T = \left(\begin{array}{ccccccc} B_0 & C_0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & B_1 & C_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & B_2 & C_2 & 0 & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & C_{k_{max}-1} \\ 0 & 0 & 0 & 0 & 0 & A_{k_{max}} & B_{k_{max}} \end{array} \right)$$

where N_{pol} is the number of Laguerre polynomials that are kept, k is the index associated with the $v_\parallel$ direction, and the A_k, B_k, C_k are square blocks of size N_{pol}. Their respective components are

$$\begin{cases} a^{(k)}_{lj} = & -\frac{\hat{N}^{jl(k)}_{1,a}}{2\Delta v_\parallel} + \frac{\hat{N}^{jl(k)}_{2,a}}{\Delta v^2_\parallel} \\ b^{(k)}_{lj} = & \hat{N}^{jl(k)}_{0,a} - 2\frac{\hat{N}^{jl(k)}_{2,a}}{\Delta v^2_\parallel} \\ c^{(k)}_{lj} = & \frac{\hat{N}^{jl(k)}_{1,a}}{2\Delta v_\parallel} + \frac{\hat{N}^{jl(k)}_{2,a}}{\Delta v^2_\parallel} \end{cases}$$

and

$$\boldsymbol{S} = \begin{pmatrix} \vdots \\ S_0^{(k)} = \kappa_{0,a}^{(k)} \hat{N}_{0,a}^{jl} + \frac{\partial \kappa_{0,a}^{(k)}}{\partial \tilde{v}_\parallel} \hat{N}_{1,a}^{jl} + \frac{\partial^2 \kappa_{0,a}^{(k)}}{\partial \tilde{v}_\parallel^2} \hat{N}_{2,a}^{jl} \\ S_1^{(k)} = \kappa_{1,a}^{(k)} \hat{N}_{1,a}^{jl} + \frac{\partial \kappa_{1,a}^{(k)}}{\partial \tilde{v}_\parallel} \hat{N}_{1,a}^{jl} + \frac{\partial^2 \kappa_{1,a}^{(k)}}{\partial \tilde{v}_\parallel^2} \hat{N}_{2,a}^{jl} \\ S_2^{(k)} = 0 \\ \vdots \\ S_{Npol-1}^{(k)} = 0 \\ \vdots \end{pmatrix}$$

The Cranck Nicholson scheme is split in the following way

$$\begin{cases} \left(I - \frac{\Delta t}{4} T^n\right) \tilde{\boldsymbol{\alpha}} & = \left(I + \frac{\Delta t}{4} T^n\right) \boldsymbol{\alpha}^n \\ \tilde{\tilde{\boldsymbol{\alpha}}} & = \tilde{\boldsymbol{\alpha}} + \Delta t \boldsymbol{S}^n \\ \left(I - \frac{\Delta t}{4} T^n\right) \boldsymbol{\alpha}^{n+1} & = \left(I + \frac{\Delta t}{4} T^n\right) \tilde{\tilde{\boldsymbol{\alpha}}} \end{cases}$$

where n stands for the time index and I is the identity matrix. The scheme is split for stability reason. Indeed, the tridiagonal by blocks inversion problem can be solved thanks to a LU decomposition valid only if the left hand side matrix is diagonal dominant. This condition gives a limit on the time step for collision as the dominant off diagonal term is proportionnal to $\frac{\Delta t}{\Delta v_\parallel^2}$. Interestingly the splitting allows for a time step twice bigger than the one without splitting.

2.6. Numerical implementation of conservation properties

Due to numerical approximations, conservation properties are not perfectly satisfied. We present here a method used to improve these conservation laws. It is used to correct only the C^1 part. Indeed the way C^0 is treated automatically satisfies conservation properties. All fluid quantities without indices correspond to the initial values. The ones noted with the prime correspond to values after the use of C^1. Finally the quantities with two primes are corrected values. The procedure is the following, in chronological order :

i)we correct the density by simply applying an homothety on the distribution function

$$F'' = \frac{F' n}{n'}$$

ii) the parallel velocity and the temperature are then corrected simultanously by removing the Maxwellian after collisions F'_M and adding a new Maxwellian F''_M with the corrected moments defined as

$$\begin{cases} V''_{\parallel a} = & V_{\parallel a} + \Delta t \sum_b \frac{R_{\parallel ab}}{n_a m_a} \\ T''_a = & T_a \end{cases}$$

The corrected parallel velocity comes from the momentum evolution equation

$$n_a m_a \frac{\partial V_{\parallel a}}{\partial t} = \sum_b R_{\parallel ab}$$

where the exchange rate of momentum is given by

$$\begin{aligned} R_{\parallel ab} & = -n_a m_a \nu_{ab} \times \\ & \left[V_{\parallel a} - V_{\parallel b} - \frac{3}{5} \frac{q_{\parallel a}}{n_a T_a} \left(\frac{1}{1+x_{ab}^2} \right) + \frac{3}{5} \frac{q_{\parallel b}}{n_b T_b} \left(\frac{1}{1+x_{ba}^2} \right) \right] \end{aligned}$$

The temperature has to be kept constant $T'' = T$

2.7. Numerical parameters

In this appendix, we detail the choice of the main numerical parameters used for the collision operator. The first step is the number of polynomials N_{pol} kept for the projection in the μ direction. For this choice, the most stringent test is to retrieve k_{neo} in the single species case. The minimal number of polynomials to have the expected poloidal rotation is $N_{pol} = 3$. Once the number of polynomials is set, one has to choose the discretization in the μ direction. A necessary condition for the projection to work properly is to ensure the orthogonality of the polynomials and so to check the condition

$$\left\| \delta_{ij} - \int du e^{-u} P_i(u) P_j(u) \right\| \ll 1 \text{ for any i,j}$$

One can show that the optimal choice for the number of points in the μ direction is $N_\mu = 64$ and the optimal value for the the upper limit in the μ direction is $\mu_{max} \simeq \frac{16T}{B}$. The number of points in the $v_\parallel$ direction is less critical in terms of numerical cost. 128 points reveal sufficient for the collision operator.

The last point is to choose the collisional time step Δt_{coll}. Indeed in order to save computational ressource, the collision operator can be used on a different time scale as the the rest of the code GYSELA. Of course, the collisional time step Δt_{coll} has to be proportional to $(\max\ (\nu_s^\star))^{-1}$where $\nu_s^\star$ is the collisionality of the s species .

3. Validation of the collision operator

To validate the collision operator, a first step is to perform conservation and relaxation tests by solving collisions only, i.e. without the effects of trajectories

$$\frac{\partial f_a}{\partial t} = \sum_b C_{ab}$$

In this section critical physical properties of the collision operator are tested : conservation properties, relaxation toward the Maxwellian and its dynamics and the exchange rates of momentum and energy between species. All the results shown here are obtained with a discretization of $(N_{v_\parallel}, N_\mu) = (128, 48)$ which is the minimal discretization for this operator.

3.1. Single species tests

Conservation laws are tested by initializing a shifted Maxwellian that belongs to the kernel of the operator and should therefore remain constant in time. After approximately 25% collision time, the following conservation are observed for an initial mach number $M_\parallel = 0.1$:

$$\frac{\Delta n}{n} \simeq 10^{-5} \qquad \Delta p_\parallel \simeq 4 \cdot 10^{-5} \qquad \frac{\Delta E}{E} \simeq 1.5 \cdot 10^{-5}$$

To investigate the dynamical relaxation to the Maxwellian, a case with $T_\parallel \neq T_\perp$ and $\frac{T_\parallel - T_\perp}{T_a} \ll 1$ is launched

$$F_a = n_a \left(\frac{m_a}{2\pi T_{\parallel a}} \right)^{1/2} \frac{m_a}{2\pi T_{\perp a}} \exp\left(-\frac{m_a v_\parallel^2}{2T_{\parallel a}} - \frac{m_a v_\perp^2}{2T_{\perp a}} \right)$$

where $T_a = \frac{T_{\parallel a} + 2T_{\perp a}}{3}$. Then at first order in $\frac{T_\parallel - T_\perp}{T_a} \ll 1$

$$f_a = 1 + \frac{T_{\parallel a} - T_{\perp a}}{3T_a} \frac{1}{v_{Ta}^2} \left(v_\parallel^2 - \frac{v_\perp^2}{2} \right)$$

Integrating $\partial_t f_a$, weighted by the energy, over the velocity space leads to :

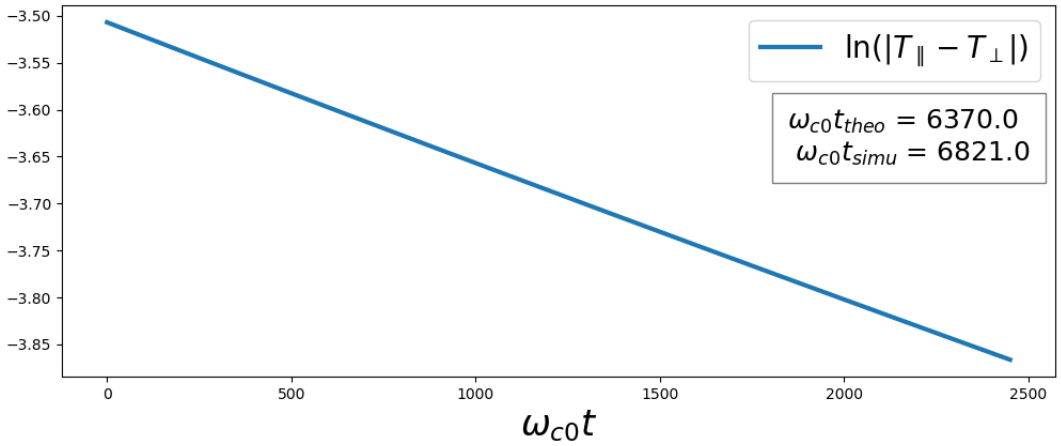
$$\frac{d \ln\left(T_\parallel - T_\perp\right)}{dt} = \frac{16}{15\sqrt{\pi}} \int_0^\infty dx e^{-x^2} x^6 \left(\nu_v + \frac{3}{2} \nu_d \right)$$

This integral can be computed either with the actual expressions of ν_v and ν_d or their approximate values :

$$\begin{cases} \frac{d \ln\left(T_\parallel - T_\perp\right)}{dt} & = -0.80 \nu_{aa} \text{ for actual expressions} \\ \frac{d \ln\left(T_\parallel - T_\perp\right)}{dt} & = -0.78 \nu_{aa} \text{ for fitted values} \end{cases}$$

The discrepancy is small, thus validating the relevance of the fitting used in the derivation of the collision operator. The prediction for the actual expressions of ν_v and ν_d is used as a theoretical prediction and compared with GYSELA results in figure 1. A mismatch of 7% percent is found. This discrepancy is acceptable as most of physics phenomena studied with gyrokinetic codes are independent of the isotropisation rate.

Figure 1: Relaxation of the parallel and perpendicular temperatures



3.2. Test with two species

The theoretical exchange rates of parallel momentum and energy between two Maxwellians are respectively

$$R_{\parallel,Mab} = -n_a m_a \nu_{ab} \left(V_{\parallel a} - V_{\parallel b}\right)$$

$$Q_{M,ab} = -3\frac{n_a m_a}{m_a + m_b}\nu_{ab}\left(T_a - T_b\right)$$

It is then easy to show that

$$\frac{d\ln\left(V_{\parallel a} - V_{\parallel b}\right)}{dt} = -\left(\nu_{ab} + \nu_{ba}\right)$$

$$\frac{d\ln\left(T_a - T_b\right)}{dt} = -2\frac{m_a\nu_{ab} + m_b\nu_{ba}}{m_a + m_b}$$

These two relations have been checked using proton and deuterium for the two species. The result for the simulation with different velocities is shown in fig.2. The result for the simulation with different temperatures is shown in fig.3. The agreement is of the order of one percent in both cases.

Figure 2: Relaxation of the parallel velocities of the two species

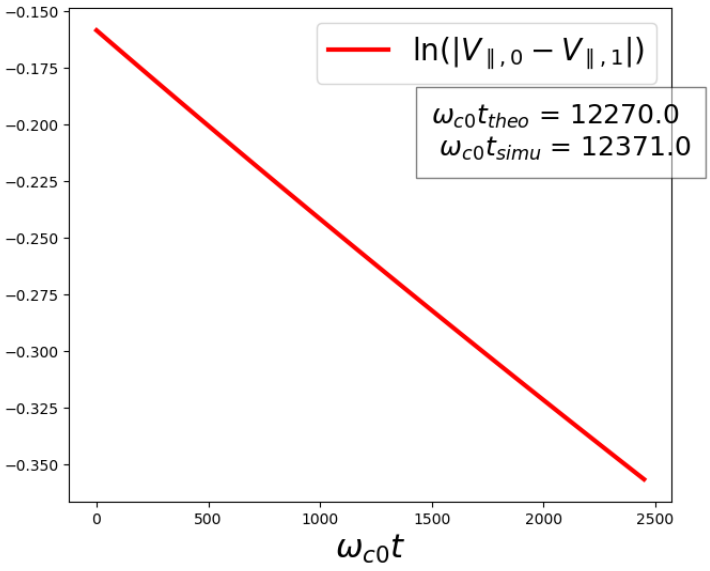
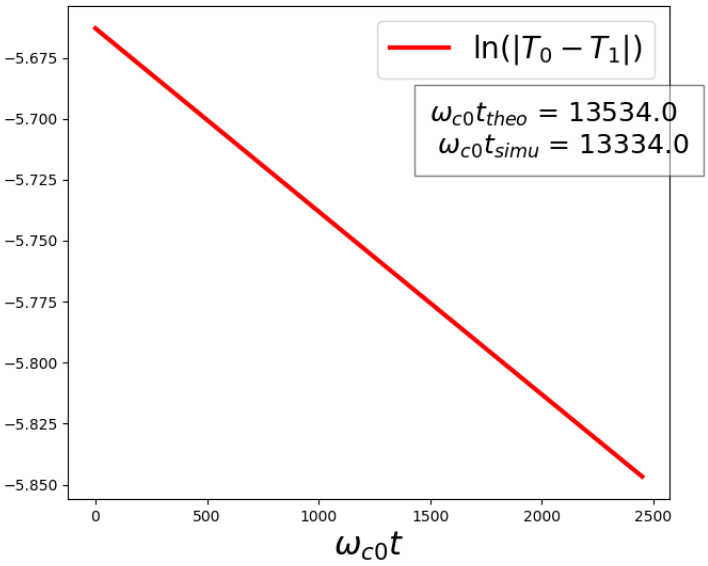


Figure 3: Relaxation of the temperatures of the two species



References

[1] P. Donnel *et al.*, Computer Physics Communications (2019), "A multi-species collisional operator for full-F global gyrokinetics codes: Numerical aspects and verification with the GYSELA code"